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NONDEGENERATE POINT PAIRS IN GLOBAL VARIATIONAL ANALYSIS

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PART I

The objectives of this paper are formulated in Part I. Definitions and theorems from earlier books and papers are organized.

1. The manifold M_n and Weierstrass integral J

This paper is concerned, in the sense of reference [12], with a Weierstrass integral J defined on a compact connected Riemannian manifold M_n. As in § 3 of [12] each locally defined 'preintegrand' F of J has values of the form

$$F(u, r) = F(u^1, \cdots, u^n; r^1, \cdots, r^n) , \qquad (n > 1) , \tag{1.1}$$

where $u = (u^1, \cdots, u^n)$ is a point of R^n in the domain U of a 'presentation' $(\varphi, U) \in \mathscr{D}M_n$ and $(r^1, \cdots, r^n) = r$ is a contravariant nonnull vector at u. F is assumed *positive definite*, *nonsingular* in the sense of § 6 of [12] and *positive regular* [12, Def. 14.2]. These are terms in classical variational theory. For references to classical variational theory see Bibliography of [12].

The Weierstrass integral J can be taken as an integral L of length on M_n, as classically defined in positive definite Riemannian geometry. In this special case the extremals are geodesics and the theorems belong to differential topology.

Extremals are studied which join a prescribed point pair $A_1 \neq A_2$ on M_n and are A_1A_2-homotopic [12, Def. 7.4] to a curve h joining A_1 to A_2 prescribed on M_n. Such extremals are intimately related to topological invariants very recently discovered by the author and termed *Fréchet numbers* $\mathbf{R}_i$. These numbers are the connectivities, over the field Q, common to the pathwise components of a basic Fréchet space $\mathscr{F}^{A_2}_{A_1}$ of "curve classes". For the original ideas of Fréchet see [3, p. 53]. The term Fréchet number was introduced by the author to distinguish the connectivities $\mathbf{R}_i$ from the different kinds of connectivities R_i appearing in the literature. Fréchet numbers are defined in [12, § 27] and [10].

We begin by recalling a simplified version of [12, Theorem 21.2].

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Theorem 1.1. *Corresponding to an arbitrary point pair $A_1 \neq A_2$ and a curve h joining A_1 to A_2 there exists at least one extremal g of J which joins A_1 to A_2, is A_1A_2-homotopic to h and affords an absolute minimum to J relative to piecewise regular curves whch join A_1 to A_2 and are A_1A_2-homotopic to h on M_n.*

The extremal g is not necessarily unique, as simple examples show. It will be termed *homotopically minimizing* when Theorem 1.1 holds. When g is given, the points A_1, A_2 are given as the endpoints of g. The point pair $A_1 \neq A_2$ will be conditioned as follows and fixed.

Definition 1.1. *A ND (nondegenerate) point pair.* Let γ be an extremal joining a point pair $A_1 \neq A_2$. Conjugate points of A_1 on γ and their multiplicities are defined in [12, § 10]. The *index* of an extremal γ joining A_1 to A_2 is the "count" of the conjugate points of A_1 on γ definitely preceding A_2, counting each conjugate point with its multiplicity. The nullity of γ is by definition the multiplicity of A_2 as a conjugate point of A_1. The extremal γ is termed *ND* if its nullity is zero.

Most importantly a point pair $A_1 \neq A_2$ is termed ND if each extremal which joins A_1 to A_2 is ND.

We consider extremals which join a *ND* point pair $A_1 \neq A_2$ and belong to a prescribed A_1A_2-homotopy class. We shall relate these extremals in a simple manner to the Fréchet numbers $\mathbf{R}_i$ of M_n. See [12, § 27].

When a point pair $P \neq Q$ is not assumed *ND*, basic extremal homology relations can still be formulated if the essential facts in the *ND* case have already been organized. [12, Theorem 27.3] is a fundamental theorem on point pairs (P, Q) of this character. We shall return to this theorem in a separate paper: *Extremal limits of ND extremals*, to appear in Rend. Mat.

The first properties of *ND* point pairs will now be recalled.

Some properties of ND point pairs (A_1, A_2). According to [12, Corollary 24.2], if (A_1, A_2) is a *ND* point pair, the number of extremals γ joining A_1 to A_2 with J-lengths less than a positive constant c is less than some finite integer N_c. N_c may become infinite with c. The number of extremals joining A_1 to A_2 with unconditioned J-length is finite or countably infinite. The key to the relations between *ND* point pairs and a degenerate point pair is the fact that the set of *ND* point pairs (A_1, A_2) is everywhere dense on the product manifold $M_n \times M_n$. This is because the set af all points on M_n conjugate to a fixed point P has the measure zero on M_n. This was first proved in [5, pp. 233–234]. In [12] see Theorem 24.1.

The following definition will facilitate the exposition.

Definition 1.2. *The set Σ_g of (J, g)-admissible extremals.* A *ND* point pair $A_1 \neq A_2$ is prescribed and a curve h joining A_1 to A_2. Theorem 1.1 is satisfied by an extremal g, A_1A_2-homotopic to h. Any extremal which joins A_1 to A_2 and is A_1A_2-homotopic to g will be called *(J, g)-admissible*. This terminology is permanent. The extremal g is fixed. Let Σ_g be the set of (J, g)-admissible extremals. The set Σ_g may be finite or countably infinite. The above extremal g will be called a *prime extremal* of J on M_n.

Definition 1.3. *The type numbers m_i^g of Σ_g.* Corresponding to each integer $i \geq 0$ let m_i^g denote the number (possibly infinite) of extremals of J of index i in the set Σ_g of (J, g)-admissible extremals. The number m_i^g is termed the ith *type number of Σ_g.*

[12, Theorem 27.1] includes the following affirmation, here termed Theorem 1.2.

Theorem 1.2. *If each of the type numbers m_i^g is finite then $m_i^g \geq \mathbf{R}_i$ for each $i \geq 0$ and*

$$
\begin{aligned}
m_0^g &\geq \mathbf{R}_0 , \\
m_1^g - m_0^g &\geq \mathbf{R}_1 - \mathbf{R}_0 , \\
m_2^g - m_1^g + m_0^g &\geq \mathbf{R}_2 - \mathbf{R}_1 + \mathbf{R}_0 , \\
\cdots\cdots &\geq \cdots\cdots .
\end{aligned}
\tag{1.2}
$$

Theorem 1.2 will not be established in this paper. However a *first* step will be taken towards proving Theorem 1.2. Under the hypothesis of Theorem 1.2 we shall here define special integers $L_i^g \geq 0$, termed *(J, g)-connectivities* of M_n. These integers are such that the inequalities of Theorem 1.2 are valid if $\mathbf{R}_i$ is replaced by L_i^g for each $i \geq 0$.

A second and final step in the proof of Theorem 1.2 will be taken in a separate paper by proving the following.

Theorem 1.3. *Under the hypotheses of Theorem* 1.2

$$L_i^g = \mathbf{R}_i . \tag{1.3}$$

The (J, g)-connectivities L_i^g of M_n are defined in § 6. They are not topological invariants a periori.

Example 1.1. If M_2 is diffeomorphic to a 2-sphere, each Fréchet number $\mathbf{R}_i = 1$. This will be shown in [2]. See also [7, Theorem 15.1, p. 247].

2. The finiteness of the type numbers m_i^g

The finiteness of the type numbers m_i^g is a condition on M_n, J and g which will be clarified by special terminology.

Definition 2.1. *Manifolds M_n which are (J, g)-finite.* In our terminology such manifolds are compact connected differentiable manifolds on which a Weierstrass integral J exists and satisfies the following two conditions:

Condition I. A homotopically minimizing extremal g of J exists whose endpoints (A_1, A_2) are a *ND* point pair $A_1 \neq A_2$.

Condition II. The resultant type numbers m_i^g are finite for the prime extremal g satisfying Condition I and for each integer $i \geq 0$.

The conditions in this definition will be better understood if M_n is considered a member of a class of manifolds now to be defined.

Definition 2.2. *The class* $((N_n))$. Let N_n, $n > 1$ be a compact connected differentiable manifold of class C^∞. Let $((N_n))$ be the class of all differentiable manifolds M_n homeomorphic to N_n.

As shown in [4] the Fréchet numbers $\mathbf{R}_i$ of all manifolds in a class $((N_n))$ are the same. For example, the numbers $\mathbf{R}_i$ are the same for an n-sphere as for an exotic sphere of Milnor type. The numbers $\mathbf{R}_i$ are the same for a classical torus in R^3 as for a flat torus.

In any given class $((N_n))$ it can be shown, by example, that there always exists a manifold M_n on which a Weierstrass integral J with extremal g exists which satisfies Condition I of Definition 2.1 but not Condition II. This statement remains true if the integrals J are required to be R-integrals. R-integrals and W-integrals are abbreviations for Riemann integrals and Weierstrass integrals. R-integrals are integrals of length. They are special W-integrals.

In this paper we are considering classes $((N_n))$ of manifolds in which there always exists a manifold M_n which is (J, g)-finite for some J and g. It is our conjecture that in every class $((N_n))$ there exists a manifold M_n which is (J, g)-finite for some J and extremal g.

In [2] it will be shown that this conjecture is true when $n = 2$. N_2 may be any abstract differentiable compact surface. A known lemma affirms that a class $((N_2))$ contains an abstract differentiable 2-manifold M_2 of constant curvature. The nullity or sign of this curvature is determined by the Euler characteristic of N_2. The manifold M_2 is taken as an "identification space". See [6, Appendix A].

If $((N_n))$ contains a manifold M_n which is (J, g)-finite for some J and g, we shall say that $((N_n))$ *is* (J, g)*-finite*. If $((N_n))$ is (J, g)-finite, the Fréchet numbers $\mathbf{R}_i$ of each manifold $M_n \in ((N_n))$ are finite, regardless of whether M_n is or is not (J, g)-finite. This is shown in [4]. For other results concerning the above conjecture see [12, § 27].

In § 3 we recall the definition and properties of elementary extremals and broken extremals which lead to our definition of the (J, g)-connectivities L_i^g of M_n.

3. Elementary extremals

The following lemma is a consequence of [12, § 19]. In this lemma a special J-length $\mathbf{m}$ is defined.

Lemma 3.1. *Corresponding to a W-integral J on M_n there exists a positive number* $\mathbf{m}$ *(termed a preferred J-length) such that the following is true:*

The extremals ξ with J-lengths $\mathbf{m}$, *issuing from a point p arbitrarily prescribed on M_n, intersect in no point other than p, bear no conjugate points of p, cover a closed topological n-disc on M_n with p an interior point and have J-lengths which afford a proper absolute minimum to J relative to "admissible" curves which join their endpoints on M_n.*

Such a lemma can be proved by relatively simple methods involving classical implicit function theory when J is the integral L of R-length. It cannot be proved so simply when J is a general W-integral. This is because geodesics are *reversible* in the sense of [12, Exercise 7.2], while extremals of W-integrals in general are not so reversible.

Note. Our mode of Proof of Lemma 3.1 in [12, § 19] makes it clear that if $\mathbf{m}$ is a "preferred length" of Lemma 3.1, then any real number in a sufficiently small neighborhood of $\mathbf{m}$ in R could also serve as a preferred length.

Definition 3.1. *Elementary extremals.* An extremal of J whose J-length is at most the preferred length $\mathbf{m}$ of Lemma 3.1 is called an *elementary extremal.*

The following definition is given in [12, § 20].

Definition 3.2. *The J-distance $\Delta(p, q)$* between points p and q on M_n is taken as the G. L. B. of J-lengths of piecewise regular curves which join p to q.

The J-distance $\Delta(p, q)$ satisfies the axioms for a distance on a metric space, except that $\Delta(p, q)$ may not equal $\Delta(q, p)$. The following theorem is a consequence of Theorem 20.1 and statement $(a)_6$ of [12, § 20]

Theorem 3.1. *The mapping*

$$(p, q) \to \Delta(p, q) \colon M_n \times M_n \to R \tag{3.1}$$

is continuous. Restricted to the subspace of $M_n \times M_n$ on which $0 < \Delta(p, q) \leq \mathbf{m}$, *the mapping* (3.1) *is of class C^∞.*

Theorem 3.1 must be supplemented by a theorem telling how a point Q on an elementary extremal ξ varies with its endpoints p, q and a parameter t on ξ. To that end let $\xi(p, q)$ be an elementary extremal on M_n with endpoints p, q such that

$$0 < \Delta(p, q) \leq \mathbf{m} \qquad (\mathbf{m} \text{ from Lemma 3.1}) . \tag{3.2}$$

Let $\hat{\xi}(p, q)$ be the extremal of length $\mathbf{m}$ with initial arc $\xi(p, q)$. For $0 \leq t \leq \mathbf{m}$ let $Q(t, p, q)$ be the point Q on $\hat{\xi}(p, q)$ such that $\Delta(p, Q) = t$. The following theorem is a consequence of [12, Theorem 20.3].

Theorem 3.2. *If Ω is the subspace of pairs $(p, q) \in M_n \times M_n$ for which* (3.2) *holds, the mapping*

$$(t, p, q) \to Q(t, p, q) : (0, \mathbf{m}] \times \Omega \to M_n \tag{3.3}$$

is of class C^∞. The extension of this mapping is continuous when $(0, \mathbf{m}]$ *in* (3.3) *is replaced by* $(0, \mathbf{m}]$.

In the next section we shall recall the definition of compact subspaces $[g]^\nu_\beta$ of the product space $(M_n)^\nu$. The spaces $[g]^\nu_\beta$ are termed *vertex spaces*. It is in terms of these vertex spaces that the limiting connectivities L^g_i can be defined when the type numbers m^g_j are finite.

4. Vertex spaces $[g]^\nu_\beta$

A *ND* point pair $A_1 \neq A_2$ and a curve h joining A_1 to A_2 have been prescribed. A_1 can be joined to A_2 by an extremal g which satisfies Theorem 1.1. The extremal g is now held fast. Let β be any value in R such that $J(g) < \beta$ and β is *J-ordinary*, that is, not the J-length of a (J, g)-admissible extremal (Def. 1.2). If ν is a positive integer such that

$$(4.1) \qquad J(g) < \beta < \mathbf{m}(\nu + 1) \qquad (\mathbf{m} \text{ from Lemma 3.1}),$$

g can be partitioned into $\nu + 1$ successive elementary extremal arcs of equal J-length $< \mathbf{m}$. The successive endpoints of these subarcs of g form a sequence

$$(4.2) \qquad A_1, p_1, \cdots, p_\nu, A_2$$

of points of M_n. The points $p_1, \cdots, p_\nu$ define a ν-tuple $\mathbf{p}$ on the ν-fold product $(M_n)^\nu$ of M_n by itself. The ν-tuple $\mathbf{p}$ is on a subspace $[g]^\nu_\beta$ of $(M_n)^\nu$ which we now define.

Definition 4.1. *A (J, g)-vertex space* $[g]^\nu_\beta$. If (4.1) holds with β J-ordinary, a maximal, pathwise connected subspace of $(M_n)^\nu$, satisfying the following three conditions is called a *(J, g)-vertex space* $[g]^\nu_\beta$.

Condition I. Each ν-tuple $z = (z_1, \cdots, z_\nu)$ of $[g]^\nu_\beta$ shall be such that successive points of M_n in the sequence

$$(4.3) \qquad A_1, z_1, \cdots, z_\nu, A_2$$

which are distinct can be joined by elementary extremals of J.

Condition II. The broken extremal, say $\zeta^\nu(z)$, joining A_1 to A_2 and defined by the successive elementary extremals joining successive distinct points in (4.3), has a J-length $\leq \beta$.

Condition III. $[g]^\nu_\beta$ contains the ν-tuple $\mathbf{p} = (p_1, \cdots, p_\nu)$ of (4.2) which partitions g into $\nu + 1$ elementary extremals of equal J-length.

That there exist (J, g)-vertex spaces is implied by the existence of the extremal g. A vertex space $[g]^\nu_\beta$ is closed in $(M_n)^\nu$ and compact. It is uniquely determined as a subspace of $(M_n)^\nu$ by J and its parameters g, ν, β. An equivalent characterization of a vertex space $[g]^\nu_\beta$ will be given in Lemma 8.1 in a larger contex. Cf. [12, Def. 24.5].

Introduction to Theorem 4.1. A vertex space $[g]^\nu_\beta$ is given. The maximal subset of extremals, which are (J, g)-admissible (Def. 1.2), have J-lengths $< \beta$ and are mutually A_1A_2-homotopic through broken extremals under the J-level β, contains the extremal g. The number of such extremals is finite according to [12, Corollary 24.2]. Let

$$(4.4) \qquad S_\beta = (\gamma_0, \cdots, \gamma_r) \qquad (\text{cf. } [12, (26.11)])$$

be the set of these extremals. Let κ be the maximum of the indices of the extremals in the set S_β. For $i = 0, 1, \cdots$, let μ_i^β be the count of extremals in the set S_β with the index i. Then [12, Theorem 26.1] with $m_i = \mu_i^\beta$ therein implies the following. (On [12, p. 201], Λ_r should be Λ_{σ_r}.)

Theorem 4.1. *Let R_i^β denote the* ith *connectivity over Q of the vertex space $[g]_\beta^\nu$. Then each R_i^β is finite, $R_0^\beta = 1$ and $R_i^\beta = 0$ for $i > \kappa$. On setting $m_i = \mu_i^\beta$ and $R_i = R_i^\beta$ the following relations hold,*

$$
\begin{aligned}
m_0 &\geq R_0 \,, \\
m_1 - m_0 &\geq R_1 - R_0 \,, \\
m_2 - m_1 + m_0 &\geq R_2 - R_1 + R_0 \,, \qquad (4.5)\\
\cdots\cdots &\geq \cdots\cdots \\
m_\kappa - m_{\kappa-1} + \cdots(-1)^\kappa m_0 &= R_\kappa - R_{\kappa-1} + \cdots(-1)^\kappa R_0
\end{aligned}
$$

implying the relations

$$\mu_i^\beta \geq R_i^\beta \qquad (i = 0, 1, \cdots) \,. \qquad (4.6)$$

The number μ_i^β will be called the *i*th *type number* of the set S_β of (J, g)-admissible extremals given in (4.4). This is in contrast with the fact that some of the global type numbers m_i^g of Definition 1.3 may be countably infinite.

Introduction to Part II. Theorem 4.1 is essential in proving Theorem 1.2. However, an extension Theorem 6.1 of Theorem 4.1, and a radical supplement, Theorem 1.3, are required to prove the ultimate Theorem 1.2. In Theoems 1.2 and 1.3 it is assumed that for a *given prime extremal* g, m_i^g is finite for each integer $i \geq 0$. This assumption is for the given prime extremal g only. Theorem 6.1 is the principal theorem of Part II.

PART II

5. The homology groups of $[g]_\beta^\nu$ and $[g]_\beta^\mu$, $\mu > \nu$

A special kind of deformation, termed a *traction*, is needed to prove the principal theorem of this section. Tractions are extensions of Borsuk's retracting deformations. See [1].

Definition 5.1. *Deformations.* Let $I = [0, 1]$ denote an interval for the time t. For us a deformation D of a subspace A of a topological space χ is a continuous mapping

$$(p, t) \to D(p, t) : A \times I \to \chi \qquad (5.1)$$

such that

$$D(p, 0) \equiv p \qquad (p \in A) \,. \qquad (5.2)$$

If F is a real-valued function with domain χ, D is called an *F-deformation* if for $(p, t) \in A \times I$

$$F(D(p, 0)) \geq F(D(p, t)) . \tag{5.3}$$

Definition 5.2. *Tractions of A into B.* Let B be a subspace of A, possibly A. The deformation D of Definition 5.1 will be called a *traction of A into B*, if D deforms A on *A into B* and deforms B *on B*. See [11, Definition 2.1].

The following lemma is proved as Lemma 2.1 of [5].

Traction Lemma 5.1. *Let a traction of A into a subspace B be given. The inclusion mapping of B into A then "induces" an isomorphic mapping of the qth homology group of B onto that of A.*

Lemma 5.1 is an extension of a classical theorem in which T is a retracting deformation of A onto B.

The principal theorem of this section is stated as follows.

Theorem 5.1. *If $[g]^{\nu}_{\beta}$ is a (J, g)-vertex space* (Def. 4.1), *then for any integer $\mu > \nu$ the vertex spaces $[g]^{\nu}_{\beta}$ and $[g]^{\mu}_{\beta}$ have isomorphic homology groups of each dimension.*

To prove Theorem 5.1 a special subspace X^{μ}_{ν} of $[g]^{\mu}_{\beta}$ will first be defined.

The subspace X^{μ}_{ν} of $[g]^{\mu}_{\beta}$. To an arbitrary ν-tuple $z = (z_1, \cdots, z_\nu) \in [g]^{\nu}_{\beta}$ a μ-tuple

$$\Theta^{\mu}_{\nu}(z) = (z_1, \cdots, z_\nu, A_2, \cdots, A_2) \in [g]^{\mu}_{\beta} \tag{5.4}$$

will be assigned, introducing $\mu - \nu$ vertices A_2. The mapping

$$z \to \Theta^{\mu}_{\nu}(z) : [g]^{\nu}_{\beta} \to [g]^{\mu}_{\beta} \tag{5.5}$$

is clearly continuous and is onto a subspace X^{μ}_{ν} of $[g]^{\mu}_{\beta}$.

Since $[g]^{\nu}_{\beta}$ and $[g]^{\mu}_{\beta}$ are compact and the mapping Θ^{μ}_{ν} a continuous, biunique mapping onto X^{μ}_{ν}, Θ^{μ}_{ν} is a homeomorphic mapping of $[g]^{\nu}_{\beta}$ *onto* X^{μ}_{ν}. The qth homology groups of $[g]^{\nu}_{\beta}$ and X^{μ}_{ν} are accordingly isomorphic. Theorem 5.1 will follow from Traction Lemma 5.1, once the following statement is proved.

(α) *There is a traction Δ of $[g]^{\mu}_{\beta}$ into X^{μ}_{ν}.*

Proof of (α). Let $y = (y_1, \cdots, y_\mu)$ be a μ-tuple of $[g]^{\mu}_{\beta}$. $\zeta^{\mu}(y)$ then denotes the broken extremal of elementary extremals joining the successive distinct points in the sequence

$$A_1, y_1, \cdots, y_\mu, A_2 \tag{5.6}$$

of points of M_n. Let

$$w = (w_1, \cdots, w_\nu) \tag{5.7}$$

be a ν-tuple of successive points on $\zeta^{\mu}(y)$ that subdivide $\zeta^{\mu}(y)$ into $\nu + 1$ subcurves of equal J-length. This length will be at most $\beta/(\nu+1)$ and so at most $\mathbf{m}$ by (4.1).

Definition of the traction Δ. Under the deformation Δ a μ-tuple $y = (y_1, \cdots, y_\mu) \in [g]^\mu_\beta$ shall have for "final" image when $t = 1$ the μ-tuple

$$(w_1, \cdots, w_\nu, A_2, \cdots, A_2) . \tag{5.8}$$

As the time t increases from 0 to 1, then under Δ the replacement, say y^t, of the μ-tuple y shall have an ith vertex that moves along $\zeta^\mu(y)$ from y_i, when $t = 0$, to the ith vertex of (5.8), when $t = 1$. Here $i = 1, 2, \cdots, \mu$.

Set $y^t = (y_1^t, \cdots, y_\mu^t)$. For $i = 1, 2, \cdots, \mu$ let $J_i(t)$ be the J-length of the subcurve of $\zeta^\mu(y)$ from A_1 to y_i^t. The rate of change of $J_i(t)$, with respect to t, shall be constant under Δ. It follows that the J-length, measured along $\zeta^\mu(y)$, between successive points in the sequence

$$A_1, y_1^t, \cdots, y_\mu^t, A_2 , \tag{5.9}$$

changes at a constant rate with respect to t. This J-length is at most $\mathbf{m}$, since this is true when $t = 0$ and $t = 1$. Hence the μ-tuple y^t is in $[g]^\mu_\beta$ for each t.

So defined Δ actually is a traction. In fact Δ deforms a μ-tuple of $[g]^\mu_\beta$ on $[g]^\mu_\beta$ into a μ-tuple (5.8), that is, into a μ-tuple in X^μ_ν. Moreover Δ deforms μ-tuples in X^μ_ν *on* X^μ_ν, as one readily sees. Thus Δ is a traction of $[g]^\mu_\beta$ *into* X^μ_ν.

It follows from Traction Lemma 5.1 that the qth homology groups of $[g]^\mu_\beta$ and X^μ_ν are isomorphic. Since the qth homology groups of X^μ_ν and $[g]^\nu_\beta$ have been proved isomorphic, Theorem 5.1 follows.

The preceding proof of Theorem 5.1 implies a theorem on the mapping Θ^μ_ν of (5.5).

Theorem 5.2. *By virtue of the chain transformation $\widehat{\Theta}^\mu_\nu$ induced by the mapping Θ^μ_ν, a j-cycle η_j on $[g]^\nu_\beta$ is bounding or nonbounding on $[g]^\nu_\beta$ according as $\widehat{\Theta}^\mu_\nu \eta_j$ is bounding or nonbounding on X^μ_ν or equivalently on $[g]^\mu_\beta$.*

This theorem follows readily on making use of the fact that Θ^μ_ν is a homeomorphic mapping of $[g]^\nu_\beta$ onto X^μ_ν and that Δ is a traction of $[g]^\mu_\beta$ into X^μ_ν. See [13, pp. 229, 230] and Traction Lemma 5.1.

6. The (J, g)-connectivities L^g_i of M_n

The method of proof of Theorem 1.2 outlined in § 1 requires that we here define an integer L^g_i for each $i \geq 0$ whenever M_n is (J, g)-finite. To do this our terminology must be extended. The extremal g remains fixed. It joins the ND pair A_1 and A_2.

Definition 6.1. *(J, g)-ordinary and (J, g)-critical values β.* A value $\beta \in R$ which is the J-length of a (J, g)-admissible extremal of index i will be called a *(J, g)-critical value* of index i. Values $\beta \in R$ which are not (J, g)-critical values will be called *(J, g)-ordinary*. According to [12, Corollary 24.2], (J, g)-critical values are isolated in R. (For g fixed)

Definition 6.2. *The connectivities* $\mathscr{R}_i^b$. Theorem 5.1 implies the following. If $[g]_b^\nu$ is a (J, g)-admissible vertex space, then for each integer $\mu > \nu$ and each integer $i \geq 0$, the ith connectivity of $[g]_b^\mu$ has a value $\mathscr{R}_i^b$ independent of μ.

The connectivities $\mathscr{R}_i^b$ are well-defined for each value $b > J(g)$ which is (J, g)-ordinary. How $\mathscr{R}_i^b$ varies for a fixed i as b increases through (J, g)-ordinary values is a question of great importance. Lemma 6.1 characterizes this behavior when M_n is (J, g)-finite. Note that $\mathscr{R}_0^b = 1$, since each vertex space is pathwise connected.

Notation for Lemma 6.1. If there are no (J, g)-admissible extremals of index i set $\pi_i = J(g)$. If, however, m_i^g is finite and positive, let π_i denote the maximum of the (J, g)-critical values of index i. In any case $\pi_i \geq J(g)$. We shall refer to the value

$$\max(\pi_i, \pi_{i+1}) = a_i \qquad (i = 0, 1, \cdots) . \tag{6.1}$$

Definition 6.3. *i-Mature values* β_i. Let an integer $i \geq 0$ be prescribed. When M_n is (J, g)-finite, a (J, g)-ordinary value β_i will be called *i-mature* if $\beta_i > a_i$ and if each (J, g)-admissible extremal of index i is A_1A_2-homotopic to g under the J-level β_i.

Lemma 6.1. *If β_i is i-mature, then the* ith *connectivity of a vertex space $[g]_{\beta_i}^\nu$ is an integer L_i^g independent of such values β_i and of integers ν such that* $\mathbf{m}(\nu + 1) > \beta_i$.

This lemma will be proved in § 8 and § 9. The integer L_i^g is thereby defined when M_n is (J, g)-finite and is called the ith (J, g)-*connectivity* of M_n. Quite independently of the lemma the 0th connectivity of a (J, g)-vertex space is 1. Thus $L_0^g = 1$. The numbers L_i^g appear in the following theorem, the principal theorem of this paper.

Theorem 6.1. *Let the manifold M_n be (J, g)-finite. Then the inequalities* (1.2) *of Theorem* 1.2 *hold if one replaces* $\mathbf{R}_i$ *by* L_i^g *for each integer* $i \geq 0$.

7. Proof of Theorem 6.1

The principal hypothesis is that M_n is (J, g)-finite (Def. 2.1). Granting the truth of Lemma 6.1, the ith connectivity of a vertex space $[g]_{\beta_i}^\nu$ is L_i^g if, for the given i, β_i is i-mature in the sense of Definition 6.3. To complete the proof of Theorem 6.1 it suffices to prove the following:

(A) If k is an arbitrary positive integer, then

$$\begin{aligned} m_0^g &\geq L_0^g , \\ m_1^g - m_0^g &\geq L_1^g - L_0^g , \\ \cdots\cdots &\geq \cdots\cdots \\ m_k^g - m_{k-1}^g + \cdots (-1)^k m_0^g &\geq L_k^g - L_{k-1}^g + \cdots (-1)^k L_0^g . \end{aligned} \tag{7.1}$$

The proof of (A) will make use of Theorem 4.1, applied to a vertex space

$[g]^\nu_\beta$. Our choice of β depends on k. Let β be any (J, g)-ordinary value such that

$$\beta > \max(\pi_0, \pi_1, \cdots, \pi_k, \pi_{k+1}) , \tag{7.2}$$

where the values π_i are defined in § 6. We require further that β be so large that each (J, g)-admissible extremal with index at most k be A_1A_2-homotopic to g under the J-lever β. Let the integer ν then be so large that $\beta < \mathbf{m}(\nu + 1)$; a (J, g)-vertex space $[g]^\nu_\beta$ then exists.

For this β, S_β of Theorem 4.1 includes the set of (J, g)-admissible extremals with indices $0, 1, \cdots, k$. Since $k \leq \kappa$ of Theorem 4.1 the first $k + 1$ relations of (4.5) hold with m_i replaced by m_i^g. By virtue of Lemma 6.1, R_i in (4.5) can be replaced by L_i^g for $i = 0, 1, \cdots, k$. The relations (4.5) thus imply the relations (7.1).

Theorem 6.1 follows once the proof of Lemma 6.1 is completed.

The proof of Lemma 6.1 begins in § 8 by recalling the definition and some of the properties of the real-valued function

$$z \to f^\nu(z) : [g]^\nu_\beta \to R \tag{7.3}$$

introduced in [12, (26.13)]. To avoid ambiguity f^ν will here be denoted by $f^{\nu,\beta}$. We shall recall the definition of f^ν in [12].

8. The real-valued function $f^\nu = f^{\nu,\beta}$

For each ν-tuple z in a (J, g)-admissible vertex space $[g]^\nu_\beta$, let $f^{\nu,\beta}(z)$ be the J-length of the broken extremal $\zeta^\nu(z)$. If b is a (J, g)-ordinary value such that

$$J(g) < b < \beta , \tag{8.1}$$

then $[g]^\nu_b$ is a subspace of $[g]^\nu_\beta$ and

$$f^{\nu,b} = f^{\nu,\beta} \mid [g]^\nu_b . \tag{8.2}$$

To more fully describe the mapping $f^{\nu,\beta}$, this mapping will be characterized as the restriction of a mapping introduced in [12, § 21] with a much larger domain. We do not abbreviate $f^{\nu,b}$ by f^ν.

Elementary broken extremals. Let $\nu > 0$ be so large an integer that

$$(\nu + 1)\mathbf{m} > \Delta(A_1, A_2) \qquad \text{(cf. [12, (21.3)])} . \tag{8.3}$$

ν-Tuples $z = (z_1, \cdots, z_\nu) \in (M_n)^\nu$ such that successive vertices in the sequence

$$A_1, z_1, \cdots, z_\nu, A_2 \tag{8.4}$$

are distinct and define elementary extremals, give rise to broken extremals $\zeta^\nu(z)$ joining A_1 to A_2 which are termed *elementary broken extremals*. The subspace of $(M_n)^\nu$ of such ν-tuples has been denoted by $Z^{(\nu)}$.

The space $Z^{(\nu)}$ has a compact closure in $(M_n)^\nu$. For $z \in Cl\, Z^{(\nu)}$ let $\mathscr{J}^\nu(z)$ denote the J-length of the broken extremal $\zeta^\nu(z)$. The mapping

$$z = \mathscr{J}^\nu(z): Cl\, Z^{(\nu)} \to R \qquad \text{(cf. [12, (21.2)])} \tag{8.5}$$

is continuous and, restricted to $Z^{(\nu)}$, of class C^∞. It is called a *vertex function.*

By virtue of [12, Theorem 21.1] the search for extremals of J which join A_1 to A_2 and which have J-lengths less than $\mathbf{m}(\nu + 1)$ is reduced to a search for critical ν-tuples of the above vertex function $\mathscr{J}^\nu$, restricted to $Z^{(\nu)}$. [12, Theorem 21.1] yields the following.

Theorem 8.1. *A necessary and sufficient condition that an elementary broken extremal $\zeta^\nu(z)$ joining A_1 to A_2 and defined by a sequence* (8.4) *be an extremal γ is that the ν-tuple z be a critical ν-tuple of the vertex function $\mathscr{J}^\nu$ restricted to $Z^{(\nu)}$.*

The extremal γ of Theorem 8.1 does not give rise to a *unique* critical ν-tuple $(z_1, \cdots, z_\nu)$ of $\mathscr{J}^\nu$. There is, however, a unique critical ν-tuple of the following type.

Definition 8.1. *J-normal ν-tuples.* A ν-tuple $z = (z_1, \cdots, z_\nu)$ of $Z^{(\nu)}$ such that the $\nu + 1$ elementary extremals of the broken extremal $\zeta^\nu(z)$ have equal J-lengths is called *J-normal.* The extremal γ of Theorem 8.1 gives rise to a unique J-normal ν-tuple z which is a critical ν-tuple of $\mathscr{J}^\nu$. Such a z is called the *J-normal ν-tuple* of γ.

The following lemma gives a basic characterization of a vertex space $[g]^\nu_\beta$. In this lemma $Cl\, Z^{(\nu)}_\beta$ denotes the subspace of ν-tuples $z \in Cl\, Z^{(\nu)}$ such that $\mathscr{J}^\nu(z) \leq \beta$. The extremal g is given as in § 1. By hypothesis $J(g) < \beta < \mathbf{m}(\nu + 1)$.

Lemma 8.1. *Let $Z^{(\nu)}$ be a subspace of $(M_n)^\nu$ of all ν-tuples $z = (z_1, \cdots, z_\nu)$ $\in (M_n)^\nu$ such that the sequences*

$$A_1, z_1, \cdots, z_\nu, A_2 \tag{8.6}$$

define "**elementary**" *broken extremals $\zeta^\nu(z)$. Then $[g]^\nu_\beta$ is that pathwise component of $Cl\, Z^{(\nu)}_\beta$ which contains the J-normal ν-tuple of the extremal g.*

Singleton extremals. The proof of Lemma 6.1 in § 9 will involve the concept of singleton extremals. An extremal γ joining A_1 to A_2 is called *singleton* if there is no other extremal joining A_1 to A_2 with the J-length of γ.

Theorem 4.1 was proved as [12, Theorem 26.1]. The first proof of this theorem was under the assumption that the (J, g)-admissible extremals of the set

$$S_\beta = (\gamma_0, \cdots, \gamma_r) \qquad \text{(see [12, (26.11)])} \tag{8.7}$$

were singleton. [12, Theorem 26.1] was then proved to be true regardless of whether the extremals in S_β were singleton or nonsingleton. The Replacement

Lemma 24.4 of [12] was essential for this purpose. For background see [9].

The proof of Lemma 6.1 in § 9 will involve a similar a priori assumption and a similar elimination of this assumption.

Under the assumption that the extremals in the set S_β of (8.7) are singleton, we suppose that the extremals in S_β are written in the order of increasing J-length. Then $\gamma_0 = g$.

9. Proof of Lemma 6.1

In the terminology of Lemma 6.1 it suffices to prove the following lemma. An integer $i \geq 0$ is given and fixed. Let $R_i X$ denote the ith connectivity, over Q, of a space X.

Lemma 9.1. *If $\beta_i < \beta$ are two (J, g)-ordinary values of which β_i is conditioned as in Lemma 6.1, then, for any integer ν such that $\mathbf{m}(\nu + 1) > \beta$,*

$$R_i[g]^\nu_\beta = R_i[g]^\nu_{\beta_i} . \tag{9.1}$$

Since (A_1, A_2) is, by hypothesis, a *ND* point pair there is (as in (4.4)) at most a finite set

$$S_\beta = (\gamma_0, \cdots, \gamma_r) \tag{9.2}$$

of (J, g)-admissible extremals with J-lengths $< \beta$, mutually A_1A_2-homotopic through broken extremals under the J-level β. By hypothesis, S_{β_i} and hence S_β, contains each (J, g)-admissible extremal of index i. Since $\beta > \beta_i >$ max (π_i, π_{i+1}) none of the extremals $\gamma_0, \cdots, \gamma_r$ with J-lengths in (β_i, β) has an index i or $i + 1$.

The truth of Lemma 9.1 is a consequence of its truth in the following two cases.

Case I. *In Case* I *there are no (J, g)-critical values in the interval (β_i, β).*

Case II. *In Case* II *there is just one (J, g)-critical value in the interval (β_i, β). The corresponding J-normal critical point of f^ν is denoted by σ.*

If there is no (J, g)-extremal other than g, Case II will never occur.

A proof of Lemma 9.1 will be given under the hypothesis that the extremals listed in (9.2) are singleton. Exactly as in the proof of Theorem 26.1 in [12] let

$$b_0 < b_1 < b_2 < \cdots < b_r \qquad \text{(cf. [12, (26.14)'])} \tag{9.3}$$

be the J-lengths of the respective (J, g)-admissible extremals listed in (9.2). For an integer ν such that $\beta < \mathbf{m}(\nu + 1)$ let

$$\tau_0, \tau_1, \cdots, \tau_r \qquad \text{(cf. [12, (26.14)''])} \tag{9.4}$$

be the J-normal ν-tuples of the respective extremals $\gamma_0, \gamma_1, \cdots, \gamma_r$. Here $r \geq 0$. The case $r = 0$ can occur.

A review of notation follows. If z is a ν-tuple in $[g]^{\nu}_{\beta}$, we have denoted by $f^{\nu,\beta}(z)$ (or simply $f^{\nu}(z)$) the J-length of the broken extremal $\zeta^{\nu}(z)$. Given $a \in R$, it is convenient to set

$$f^{\nu}_{a} = \{z \in [g]^{\nu}_{\beta} \mid f^{\nu}(z) \leq a\} .$$

In particular $f^{\nu}_{\beta} = [g]^{\nu}_{\beta}$.

Proof in Case I. Theorem 1 of Appendix IV, [12], was proved first when $r > 0$. The deformation θ_e in this theorem is an f^{ν}-deformation of f^{ν}_{β}. In Case I it yields an f^{ν}-traction of f^{ν}_{β} into $f^{\nu}_{\beta_i}$, at least if the parameter e of θ_e is sufficiently small. In case $r = 0$ one infers an f^{ν}-traction of f^{ν}_{β} into $f^{\nu}_{\beta_i}$ from Theorem 1a of Appendix IV of [12]. (9.1) follows. See page 241.

Proof in Case II. We shall apply [5, Corollary 5.1] to f^{ν} in place of F. The above critical point σ of f^{ν} has, by hypothesis, an index k which is neither i nor $i + 1$. It follows from [5, Corollary 5.1] that

$$R_i\, f^{\nu}_{\beta} = R_i\, f^{\nu}_{c} \qquad (\beta_i < f^{\nu}(\sigma) < \beta)$$

for some value c in the interval (β_i, b) where $b = f^{\nu}(\sigma)$.

Now c is a (J, g)-ordinary value $> \beta_i$ and there is, by hypothesis, no (J, g)-critical value in the interval (β_i, c). Hence by Lemma 9.1, as established in Case I,

$$R_i\, f^{\nu}_{c} = R_i\, f^{\nu}_{\beta_i} .$$

We infer then in Case II that

$$R_i\, f^{\nu}_{\beta} = R_i\, f^{\nu}_{\beta_i}$$

or, equivalently, that (9.1) is true in Case II.

Lemma 9.1 follows when $\gamma_0, \cdots, \gamma_r$ are singleton.

The relation (9.1) is true even when some of the extremals γ_i of S_{β} fail to be singleton. A clear proof of this fact requires much more detail. Reference [9] gives some of the details when J is a Riemannian integral of length. Reference [9] will be supplemented by a similar but more complete treatment of Weierstrass integrals in the nonsingleton case. Cf. Replacement Lemma 24.4 of [12].

Granting the truth of Lemma 9.1 in the general case, singleton or nonsingleton, Lemma 6.1 follows as well as Theorem 6.1. Theorem 6.1 is the first step in the proof of Theorem 1.2. The second step, a proof of Theorem 1.3 will follow in a separate paper.

We shall add a lemma needed in the proof of Theorem 1.3.

Lemma 9.2. *Under the hypothesis that the manifold M_n is (J, g)-finite, let $[g]^{\nu}_{\beta_i}$ and $[g]^{\nu}_{\beta}$ be vertex spaces with $\beta > \beta_i$ and β_i conditioned as in Lemma* 6.1.

A prebase of singular i-cycles for the ith *homology group, over* Q, *of* $[g]^{\nu}_{\beta_i}$ *is a prebase for the* ith *homology group of* $[g]^{\nu}_{\beta}$.

By a *prebase* for a singular homology group H_i, over Q, of finite dimension is meant a set of singular i-cycles which includes just one i-cycle from each homology class of a base for H_i.

The lemma is trivially true if $i = 0$, since the space $[g]^{\nu}_{\beta_i} \subset [g]^{\nu}_{\beta}$ and both spaces are pathwise connected. Suppose then that $i > 0$. We refer to Case I and Case II, as introduced in the proof of Lemma 9.1.

Proof in Case I. As indicated in the proof of Lemma 9.1 in Case I, there exists a traction of $[g]^{\nu}_{\beta}$ into $[g]^{\nu}_{\beta_i}$. Lemma 9.2 follows from Traction Lemma 5.1.

Proof in Case II. We refer to the mapping f^{ν} of $[g]^{\nu}_{\beta}$ into R introduced in § 8. Let σ be the J-normal critical point of f^{ν} and b the critical value $f^{\nu}(\sigma)$ introduced in the proof of Lemma 9.1. By hypothesis of Case II, b is the only critical value of f^{ν} on the interval (β_i, β) and the index of σ, say k, is neither i or $i + 1$. We identify $f^{\nu}, \beta, \beta_i, b, \sigma$ respectively, with F, β, c, a, σ of [5, § 1]. By hypothesis k is the index of σ, and $i \neq k$ or $k - 1$. Suppose first that $k > 0$.

We refer to the five subsets of F_{β} listed in [5, (1.14)] of which the first is F_{β} and the last F_c. Let

$$H_i^{(1)}, H_i^{(2)}, H_i^{(3)}, H_i^{(4)}, H_i^{(5)}\ , \tag{9.5}$$

be the ith homology groups over Q if the respective sets listed in [5, (1.14)]. To verify Lemma 9.2 in Case II it suffices to prove the following.

(α) *A prebase of each of the five homology groups* $H_i^{(\mu)}$ *of* (9.5), *except the first, is a prebase of the preceding homology group.*

That (α) is true follows when $\mu = 5$ from [5, Lemma 1.2], It is true when $\mu = 4$ by [5, Proposition 3.3 (1)], since q (taken as i) is neither k nor $k - 1$. It is true when $\mu = 3$ by virtue of [5, Lemma 1.1]. Its truth when $\mu = 2$ follows from the existence of the appropriate Traction Theorems of Appendix IV of [12]. This completes the proof in Case II when $r > 0$.

The case $k = 0$. This is a subcase of Case II in which the extremal, say γ_j, in S_{β} with J-length in (β_i, β), has the index $k = 0$. Let σ be the J-normal ν-tuple of γ_j. $f^{\nu}(\sigma)$ equals a critical value b_j listed in (9.3) with $j > 0$. By hypothesis $i > 0$ in Lemma 9.2. We shall apply Traction Theorem Ω_j in [12, Appendix IV]. Let $\mu = (n - 1)\nu$. When $k = 0$ the set λ_j in Traction Theorem Ω_j is a topological μ-ball of ν-tuples on $(M_n)^{\nu}$, a ball which tends to 0 in diameter with its parameter e_j and on which f^{ν} has an absolute minimum b_j. The Traction Theorem Ω_j implies the following.

(β) *For some value* $c_{j-1} \in (b_{j-1}, b_j)$ *and for a sufficiently small* λ_j *there exists an* f^{ν}*-traction of* f^{ν}_{β} *into* $\lambda_j \cup f^{\nu}_{c_{j-1}}$.

From Traction Lemma 5.1 it follows that the ith homology group of f^{ν}_{β} is isomorphic to the ith homology group of $\lambda_j \cup f^{\nu}_{c_{j-1}}$ and hence of $f^{\nu}_{c_{j-1}}$.

Lemma 9.2 follows.

In case the critical values of j^ν on the interval (β_i, β) are singleton, the truth of Lemma 9.2 is an obvious consequence of its truth in Case I and Case II. The truth of Lemma 9.2 when some of the critical values of f^ν on the interval (β_i, β) fail to be singleton will be made clear by a paper on singleton extremals of a Weierstrass integral.

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